HYPERSTABILITY

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ABSTRACT

This paper describes a closed form demographic model with changing vital rates. The hyperstable model replaces the strict stable population assumption of constant rates of fertility and mortality with weaker assumptions on the pattern of net maternity. Those alternative assumptions are a fixed proportional distribution of births by age of mother, a fixed proportional distribution of net maternity, and a constant generation length.

In continuous form, we examine 6 different birth functions that span a broad range of demographic experience, illustrating cubic exponential and sinusoidal models with hypothetical data. In discrete form, assuming a fixed proportional distribution of births, we demonstrate that convergence to hyperstability follows the same strong ergodic theorem as convergence to stability. Data for the United States 1960-89 show that hyperstable estimates are close to observed Net Reproduction Rates, even though the post-1960 U.S. population is not hyperstable. Analytically, hyperstable relationships can model alternative trajectories to stationarity. The flexible and dynamic nature of hyperstable populations suggests great potential for use in demographic modeling and estimation.

Hyperstability

The stable population has long been the principal model of mathematical demography. Its strengths come from its logical closure, its ability to reflect the implications of any set of birth and death rates, and its central insight that constant vital rates lead to exponential growth and an unchanging age distribution. Yet the stable model has a great weakness---it is based on unchanging vital rates. That fixed rate requirement essentially makes the stable population a static model, unable to reflect the behavior of actual populations.

Dynamic models, with changing vital rates, are needed to enable population analysis to capture observed demographic behavior. Despite the research done to date, progress toward dynamic modeling has been impeded by a fundamental problem. With the exception of some limited, special cases, there is no known closed form relationship for the sequence of births generated by a series of changing vital rates.

This paper addresses that problem. It extends stable population theory to develop what we have termed hyperstable models, populations governed by closed form mathematical equations that can reflect a large number of birth sequences and, in each case, specify consistent sequences of vital rates. We explore the structure of hyperstable models, discuss some of the relationships that exist within them, demonstrate convergence to

hyperstability, and offer several numerical illustrations.

A BRIEF LOOK AT THE LITERATURE ON CHANGING RATES

A milestone in the work on the deterministic [1] modeling of dynamic rates is the monograph by Coale (1972), which contained in-depth analyses of the relationship between changing vital rates and the birth sequence. In an analysis particularly relevant here, Coale (1972, Ch. 4) investigated the assumption that, for some constant k,

$$m(x,t) = e^{kt}\, m(x,0) \qquad (1)$$

where m(x,t) is the fertility rate at age x and time t. He found that the resultant birth sequence was approximated by

$$B(t) \;.\; B(0)\exp[\tfrac{1}{2}kt + \tfrac{1}{2}kt^2/T_0]$$

where B(t) is the number of births at time t and T_0 is the mean age at childbearing at the time when the population's Net Reproduction Rate (NRR) equals one. Coale also presented a closer approximation involving higher powers of t, but could not find a closed form expression for B(t).

Cyclically stable populations, which result when a fixed sequence of rate schedules repeats itself indefinitely, provide another approach to analyzing changing rates (cf. Coale, 1972, Ch. 6; Caswell, 1989, Ch. 8; Tuljapurkar, 1985). An explicit solution can be given for a sequence of 2 rate schedules in a

population with 2 reproductive age groups (Schoen and Kim, 1994), but most cyclical populations are far too complex for direct algebraic solution.

The general case of a population whose vital rates are free to change over time is even more complex. Coale (1972, Ch. 7) used Fourier analysis to link fertility schedules and their resulting birth sequences. While a valuable advance, his solution was quite complex and led him to conclude (p208), "the general case will always remain merely calculable, but not readily understandable at an intuitive level". Kim (1987), through an analysis of the discrete product matrix, found a general algebraic solution that connected changing rates and their birth sequences. Again, in most instances, her solution is too complex to render in closed form.

Another related line of research focuses on "inverse projection" and "back projection" (cf. Lee, 1974, 1985; Oeppen, 1993; Wachter, 1986). Generally applied to studies in historical demography, inverse and back projection can be used to estimate a population's past age composition and levels of fertility and mortality from data on the total number of births and deaths. That approach shares the current interest in the long term relationship between births and fertility levels. However, it typically assumes a fixed age pattern of fertility and mortality and seeks numerical results that fit a given set of data, rather than setting forth a closed form model and examining

relationships within that model.

Nonlinear models, where the vital rates receive feedback from the birth sequence, have been another area of research involving dynamic rates. Samuelson (1976) found an explicit solution for one particularly simple case, but analyses of nonlinear models typically encounter great complexity and can lead to "chaotic" behavior (cf. May, 1974; Wachter, 1991). A recent review of the area noted the continuing difficulty in relating birth sequences in nonlinear models to their underlying mortality and fertility behavior (Wachter, 1993).

SPECIFYING THE CONTINUOUS HYPERSTABLE MODEL

Given the formidable obstacles to finding the birth sequence generated by arbitrarily changing vital rates, our approach is to start with a specified birth sequence and look for consistent series of vital rates. We begin with Lotka's classic renewal formulation, rewritten to allow the vital rates to change over time. When the population initiating the renewal process does not contribute to births at time t, we can write

$$B(t) = \int_{0}^{\infty} B(t-x)p(x,t)m(x,t)dx \qquad (2)$$

where p(x,t) is the cohort probability of surviving to age x at time t from birth at time t -x. For convenience, we introduce the net maternity function $\phi(x,t)$, where $\phi(x,t)=p(x,t)m(x,t)$, and the birth function g(t), where

$$B(t) = B(0)g(t) \qquad (3)$$

In demographic applications, g(t) should be nonnegative and we must have g(0)=1. Using equation (3), we can rewrite equation (2) as

$$1 = \int_{\alpha}^{\beta} [g(t-x)/g(t)]\ \phi(x,t)\ dx \qquad (4)$$

Each age's proportional contribution to births at time t, f(x,t), is thus

$$f(x,t) = [g(t-x)/g(t)]\ \phi(x,t) \qquad (5)$$

and net maternity at age x and time t can thus be expressed in terms of the birth sequence and f(x,t). There are an infinite number of ways to do so, but we focus on three of them.

Assuming a fixed proportional distribution of births (f)

If each age's proportional contribution to the total number of births does not vary over time, i.e. if

$$f(x,t) = f(x) \qquad (6)$$

then equation (5) can be rewritten

$$\phi(x,t) = f(x)\ [g(t)/g(t-x)] \qquad (7)$$

A constant proportional distribution of births by age of mother is a demographically plausible assumption implicit in all stable populations. Under that "fixed f" assumption, equation (7) provides an age-time array of net maternity values consistent

with g(t). The result is that equations (4) and (7) specify a hyperstable model which replaces the stable population requirement of constant mortality and fertility with the much looser requirement of a constant proportional distribution of births by age of mother.

Assuming an Unchanging Age Pattern of Net Maternity (ϕ)

A second plausible assumption is that the age pattern of net maternity does not change, though its level varies over time. Indeed, proportional change at all ages is frequently assumed because of the regularity of age patterns of net maternity. Mathematically, we can write

$$\phi(x,t) = k(t)\ \phi(x,s) \qquad (8)$$

where k(t) adjusts a base (or standard) net maternity $\phi(x,s)$ to yield the requisite birth sequence. Substituting equation (8) into equation (5) and integrating, using the fact that $*f(x,t)=1$, yields

$$k(t) = 1 \Big/ \int_{\alpha}^{\beta} [g(t-x)/g(t)]\ \phi(x,s)\, dx$$

and hence

$$\phi(x,t) = \phi(x,s) \Big/ \int_{\alpha}^{\beta} [g(t-x)/g(t)]\ \phi(x,s)\, dx \qquad (9)$$

Equation (9) provides an array of net maternity rates consistent with a given birth sequence under the assumption that the proportional distribution of the ϕ rates is constant. Although the "proportional ϕ" assumption may seem simpler than the "fixed f"

assumption, it requires integration and usually leads to a more complicated expression for net maternity.

Assuming a Constant Length of Generation (A)

A third approach is to apply the Mean Value Theorem to equation (4) and write

$$1 = [g(t - A(t))/g(t)] \int \phi(x,t)dx$$

where A(t) denotes the appropriate mean age at time t. If we assume the mean age is constant over time, we can write

$$R_0(t) = g(t)/g(t - A) \qquad (10)$$

where $R_0(t) = \int \phi(x,t)dx$ represents the NRR at time t. Mean age A is the length of time required for births to grow by a factor equal to their NRR, and thus can be viewed as a measure of generation length. Equation (10) provides a simple analytical relationship between the birth sequence and the level of net maternity. It does not specify a schedule of net maternity rates, but the Mean Value Theorem insures that such a schedule must exist [2].

Let us now define $\rho(t)$, the hyperstable growth rate of births at time t, by

$$\rho(t) = (d/dt) \ln g(t) \qquad (11)$$

When the population initiating the renewal process does not contribute to births at time 0, we can use the fact that g(0)=1

to rewrite equation (11) as

$$g(t) = \exp\left[\int_0^t \rho(y)dy \right] \qquad (12)$$

From equation (10), it follows that

$$R_0(t) = \exp\left[\int_{t-A}^t \rho(y)dy \right] \qquad (13)$$

In general, equation (12) shows how the birth sequence is generated by $\rho(t)$. Under the "constant A" assumption, equation (13) shows that the NRR sequence is also readily generated by the hyperstable growth rate of births.

HYPERSTABILITY WITH SELECTED FORMS OF g(t)

The birth sequence g(t) can take many forms. Let us consider some specific examples and their associated hyperstable net maternity functions under the fixed f assumption.

a. The Basic Exponential (Stable) Form

Let us begin with the case where births are growing exponentially, i.e. where

$$g(t) = \exp[h_1 t] \qquad (14)$$

Then, from equation (7) and the continuing assumption that the population initiating the renewal process does not contribute to births at time 0, net maternity is given by

$$\phi(x,t) = \phi(x,0) \qquad (15)$$

and, from equation (4), those constant rates yield the

conventional stable population renewal (or characteristic) equation

$$1 = \int_{\alpha}^{\beta} \exp[-h_1 x]\ \phi(x,0)dx \qquad (16)$$

where h_1 is Lotka's r, the intrinsic rate of natural increase. Stability is thus a special case of hyperstability.

b. The Quadratic Exponential Form

Let the birth function be

$$g(t) = \exp[h_1 t + h_2 t^2] \qquad (17)$$

Then the net maternity function is

$$\phi(x,t) = \phi(x,0)\ \exp[2h_2 xt] \qquad (18)$$

and the hyperstable characteristic equation is

$$1 = \int_{\alpha}^{\beta} \exp[-h_1 x + h_2 x^2]\ \phi(x,0)dx \qquad (19)$$

Equations (17)-(19) show that when fertility increases exponentially over both age and time, the birth sequence is an exponentiated quadratic. Unlike the e^{kt} pattern of change used by Coale (1972) and shown in equation (1), the e^{kxt} pattern of equation (18) is associated with a simple closed form expression for the birth sequence.

The quadratic hyperstable model has potential applications to many analyses where the level of net maternity can be thought of as monotonically increasing or decreasing over time. That

usefulness is enhanced because equation (18) describes a substantively meaningful pattern of fertility change. A decrease in net maternity over time (i.e. $h_2<0$) implies a greater decline in net maternity at higher ages relative to lower ages. That is consistent with the typical pattern of fertility decline and with the greater concentration of higher order births at older ages.

The parameters of the hyperstable model must satisfy only one constraint, that imposed by the characteristic equation. Thus when the time 0 net maternity schedule and h_2, its rate of exponential change, are known, h_1 can be found from equation (19). The value of h_1 is thus contingent on the value of h_2; h_1 is not the intrinsic growth rate implied by the $\phi(x,0)$ unless $h_2=0$.

Because both the birth sequence and the net maternity rates change in relatively simple ways, the structure of the quadratic hyperstable model can be explicated in some depth. However, that task is beyond the scope of the present paper.

c. The Cubic Exponential Form

When the birth function is

$$g(t) = \exp[h_1 t + h_2 t^2 + h_3 t^3] \qquad (20)$$

the net maternity function is

$$\phi(x,t) = \phi(x,0) \exp[2h_2 xt + 3h_3 xt(t!x)] \qquad (21)$$

and the hyperstable characteristic equation is

$$1 = \int_{\alpha}^{\beta} \exp[\,-h_1 x + h_2 x^2 - h_3 x^3] \; \phi(x,0)\,dx \qquad (22)$$

The cubic exponential form can model birth sequences that exhibit 2 changes in direction, and thus has the potential for a wide range of applications. The pattern of net maternity change indicated by equation (21) is not easy to visualize, but we find that it yields realistic age patterns over time. For example, Figure 1A shows $\phi(x,t)$ curves for 4 time points based on the net maternity pattern observed [3] in the United States, 1959-61. The age curve of net maternity exhibits a reasonable pattern at every time point, showing a slight skew to the right that increases as net maternity declines.

d. The General Exponentiated Polynomial Form

Let us now consider birth functions of the general form

$$g(t) = \exp\left[\sum_{j=1}^{n} h_j t^j\right] \qquad (23)$$

where n is the order of the polynomial chosen. The associated net maternity function is

$$\phi(x,t) = \phi(x,0)\,\exp\left[\sum_{j=1}^{n} h_j \{t^j + (-x)^j - (t-x)^j\}\right] \qquad (24)$$

and the hyperstable characteristic equation is

$$1 = \int_{\alpha}^{\beta} \exp\left[\sum_{j=1}^{n} h_j (-x)^j\right] \phi(x,0)\,dx \qquad (25)$$

With an arbitrary number of parameters, this functional form can model virtually any birth sequence.

e. The Sinusoidal Form

Hyperstable populations can take on a number of cyclically repeating forms. Let us consider the birth function

$$g(t) = 1 + b \sin(2\pi t/T) \qquad (26)$$

where b is the amplitude of the sine wave (*b*<1), and T is the period of cyclicity. The net maternity function is then given by

$$\phi(x,t) = \phi(x,0)[1+b \sin(2\pi t/T)][1 - b \sin(2\pi x/T)]/[1+b \sin(2\pi\{t-x\}/T)] \qquad (27)$$

and the hyperstable characteristic equation is

$$1 = \int_{\alpha}^{\beta} [1 - b \sin(2\pi x/T)] \phi(x,0)dx \qquad (28)$$

Figure 1B shows net maternity functions at 4 time points for a sinusoidal hyperstable population with b=.2 and T=60 years. Despite the complexity of equation (27), those $\phi(x,t)$ curves show a conventional pattern and change in a reasonable manner. (The unimodal pattern can break down at very short cycle lengths, however. Marc Artzrouni has pointed out that with b=.32 and T=9.5 years, the ϕ curve becomes trimodal.)

f. The Double Exponential Form

If the rate of change in births is exponential, i.e. if

$$\rho(t) = ce^{-bt} \qquad (29)$$

then

$$g(t) = \exp[\ (c/b)(1 \ ! \ e^{!bt})\ ] \qquad (30)$$

the net maternity function is given by

$$\phi(x,t) = \phi(x,0)\ \exp[\ (c/b)(1 \ ! \ e^{!bt})(1 \ ! \ e^{bx})\ ] \qquad (31)$$

and the hyperstable characteristic equation is

$$1 = \int_{\alpha}^{\omega} \exp[\ (c/b)(1 \ ! \ e^{bx})\ ]\ \phi(x,0)dx \qquad (32)$$

As shown in a later section, this double exponential form can model a population's transition to zero growth.

THE DISCRETE HYPERSTABLE MODEL

Specifying the Discrete Model

Under the fixed f assumption, the hyperstable model can be written in discrete time using a conventional population projection matrix (PPM) approach. The discrete form facilitates applications to data and provides the basis for analyzing convergence to hyperstability. The following discussion builds on the treatments of PPMs and convergence in Keyfitz (1977), Kim (1987), Caswell (1989), and Kim and Schoen (1993).

At time t, let $\mathbf{A}_t$ be the (n x n) PPM of a hyperstable population whose discrete characteristic equation is

$$1 = \sum_j \lambda_t^{!j} p_{tj} b_{tj} \qquad (33)$$

where λ_t is the maximal eigenvalue of $\mathbf{A}_t$, p_{tj} is the probability of surviving from the first age group at time t !j to the jth age group at time t !1, and b_{tj} is the "fertility" rate applied to the

number of persons in the jth age group at time $t-1$ to yield that group's contribution to the first age group at time t. Let $\mathbf{x}_t$ be a vector whose jth element, x_{tj}, is the number of persons in the jth age group of the hyperstable population at time t. We then have the hyperstable population projection relationship

$$\mathbf{x}_t = \mathbf{A}_t\mathbf{x}_{t-1} \tag{34}$$

To make use of the fixed f assumption, let f_j be the fixed proportion of births contributed by persons in the jth age group. Then $\Sigma_j f_j = 1$, and at any time t we can write

$$f_j = b_{tj}x_{t-1,j}/x_{t1} \tag{35}$$

Using equation (35), we can write the nonzero elements of PPM $\mathbf{A}_t$ as

$$\mathbf{A}_t = \begin{bmatrix} f_1x_{t1}/x_{t-1,1} & f_2x_{t1}/x_{t-1,2} & \cdots & f_nx_{t1}/x_{t-1,n} \\ s_{t1} & & & \\ & s_{t2} & & \\ & & \ddots & \\ & & & s_{t,n-1} \end{bmatrix} \tag{36}$$

where subdiagonal element s_{tj} is the probability of survival from age j at time $t-1$ to age $j+1$ at time t (i.e. $p_{tj} = s_{t,j-1}s_{t-1,j-2}\ldots s_{t-j+1,1}$).

Now let us define $\mathbf{F}$ as an (n x n) matrix which has f_j as the jth element in the first row, ones in the subdiagonal, and zeros elsewhere. Then $\mathbf{F}$ has the form of a PPM, and from equation (33)

its maximal eigenvalue is 1. With $\mathbf{X}_t$ an (n x n) diagonal matrix whose jth diagonal element is x_{tj}, we can write PPM $\mathbf{A}_t$ in the form

$$\mathbf{A}_t = \mathbf{X}_t \mathbf{F} \mathbf{X}_{t-1}^{-1} \tag{37}$$

as $s_{t,j-1} = x_{tj}/x_{t-1,j-1}$. The form of equation (37) is reminiscent of a similarity transformation, but the subscripts of $\mathbf{X}$ are not identical. Equation (37) is an important relationship because it shows that any fixed f PPM can be decomposed into time dependent factors reflecting the hyperstable age distribution and a time invariant matrix reflecting the f_j.

With $\mathbf{M}(t,t+w)$ denoting the product matrix $\mathbf{A}_{t+w}\mathbf{A}_{t+w-1}...\mathbf{A}_{t+1}$, we have

$$\mathbf{x}_{t+w} = \mathbf{M}(t,t+w)\,\mathbf{x}_t \tag{38}$$

and, from equation (37),

$$\mathbf{M}(t,t+w) = \mathbf{X}_{t+w}\,\mathbf{F}^w\,\mathbf{X}_t^{-1} \tag{39}$$

Matrix $\mathbf{F}$, which embodies the fixed proportional distribution of births, clearly plays a key role in hyperstable dynamics.

Convergence to Hyperstability

In general, we agree with Coale (1972, p119) that "If we can find an analytical expression for the birth sequence [that results from a specified net maternity pattern] we can be sure that it is indeed the birth sequence that would be approached ... because of the proven tendency for the effect of initial conditions to vanish with the passage of time." Under the fixed f

assumption, we can go further and demonstrate mathematically how convergence to hyperstability occurs.

Assume that a specified population adopts a given regime of hyperstable vital rates beginning with PPM $\mathbf{A}_{s+1}$, and that $\mathbf{y}_t$ is the vector whose jth element, y_{tj}, is the number of persons in that population's jth age group at time t. We can then write

$$\mathbf{y}_{s+w} = \mathbf{A}_{s+w}\mathbf{A}_{s+w!1}...\mathbf{A}_{s+1}\mathbf{y}_s = \mathbf{M}(s,s+w)\,\mathbf{y}_s \qquad (40)$$

The relative age distribution at time s+w is independent of the initial age distribution if $\mathbf{M}(s,s+w)$ is of rank 1 (Caswell, 1989, Ch. 8). When w is large, $\mathbf{M}(s,s+w)$ becomes a matrix of rank one by virtue of weak ergodicity. Population vector $\mathbf{y}_{s+w}$ has the hyperstable age distributionas the time s age distribution has been "forgotten".

Assume that matrix $\mathbf{F}$ is primitive, a conventional assumption that holds if any 2 elements in the first row, say f_i and f_j, are greater than zero and the greatest common divisor of i and j is 1. Matrix $\mathbf{F}^w$ must become a matrix of rank one when w is large because of strong ergodicity, the process that insures convergence to stability (cf. Caswell, 1989). Because the maximal eigenvalue of $\mathbf{F}$ is 1, $\mathbf{F}^w = \mathbf{u}_f\,\mathbf{v}'_f$, where $\mathbf{u}_f$ and $\mathbf{v}_f$ are the right and left eigenvectors of $\mathbf{F}$ normalized so that $\mathbf{v}'_f\,\mathbf{u}_f = 1$. Their jth elements are $u_{fj} = 1$, and $v_{fj} = (\sum^n_{i=j} f_i)/\mu$, where the hyperstable mean age at childbearing $\mu = \sum_j jf_j$.

With $\mathbf{F}^w$ a matrix of rank one, the product matrix $\mathbf{M}(s,s+w)$ becomes, from equation (39),

$$\mathbf{M}(s,s+w) = (\mathbf{X}_{s+w}\mathbf{u})(\mathbf{v}'\mathbf{X}_s^{!1}) \qquad (41)$$

also a matrix of rank one. Thus $\mathbf{M}$ becomes rank one ("converges") when $\mathbf{F}^w$ becomes rank one, and convergence to hyperstability parallels the convergence to stability of a population subject to unchanging PPM $\mathbf{F}$. That is a significant finding because it not only shows that under the fixed f assumption the age composition of a population subject to a regime of hyperstable vital rates becomes hyperstable, but indicates that exactly the same process brings about convergence in both stable and hyperstable models.

With $\mathbf{M}(s,s+w)$ a rank one matrix, we can rewrite equation (41) as a scalar times a product of vectors, i.e.

$$\mathbf{M}(s,s+w) = (x_{s+w,1}/x_{s1})\ \mathbf{u}_{s+w}\mathbf{v}'_s \qquad (42)$$

Vector $\mathbf{u}_{s+w}$, whose jth element is $u_{s+w,j} = x_{s+w,j}/x_{s+w,1}$, provides the relative hyperstable age distribution at time s+w. Vector $\mathbf{v}_s$, whose jth element is $v_{sj} = v_{fj}x_{s1}/x_{s!j+1,1}$, is a fixed vector of reproductive values. As shown by Kim and Sykes (1976), the age distribution depends only on recent rates while the reproductive values depend only on early rates. From equations (40) and (42), the hyperstable population at time s+w is completely specified by

$$\mathbf{y}_{s+w} = Q_s(x_{s+w,1}/x_{s1})\ \mathbf{u}_{s+w} \qquad (43)$$

where $Q_s = (\mathbf{v}'_s \mathbf{y}_s)$ is the total population reproductive value at initial time s.

ILLUSTRATIVE HYPERSTABLE CALCULATIONS

Patterns in 2 Hypothetical Hyperstable Populations

Let us examine the demographic behavior of some illustrative cubic and sinusoidal hyperstable populations. For the hypothetical cubic model whose age-specific net maternity rates are shown in Figure 1A, Figure 2A shows the g(t) function and the NRRs associated with our 3 alternative approaches for relating the g(t) and $\phi(x,t)$ functions. Over the 45 year time period shown, the birth function first rises gradually and then falls somewhat, while the reproductive level declines steadily. The 3 associated NRRs are virtually indistinguishable. Because cubic coefficient h_3 is positive, the population will eventually exhibit monotonically increasing birth levels and NRRs.

Figure 2B depicts the same demographic functions for the sinusoidal hyperstable population of Figure 1B. While g(t) is a true sine curve, the 3 NRR curves are only approximately sinusoidal. Relative to g(t), the NRR curves have a larger amplitude and a perceptible phase shift, even though their pattern remains the same from cycle to cycle. The fixed f and proportional ϕ approaches yield very similar NRRs, while the NRRs from the constant A approach show somewhat greater variability at the extremes.

Figure 3A shows the age distributions for ages 0 through 50 for the cubic hyperstable population of Figures 1A and 2A, using mortality for U.S. females, 1960. Because of the nature of the hyperstable parameters, the number of births rises in the years before t=0. Thus the hyperstable population becomes older between t=0 and t=45, despite the relative flatness in the birth sequence during that interval. The age distribution appears to pivot at a point around age 23, although the exact pivot point increases slightly over time. Taken together, Figures 1A, 2A, and 3A depict the dynamics of a demographic change: as an initially growing population experiences a period of fertility decline, its birth sequence peaks and falls, and population aging proceeds.

Figure 3B presents age distributions for the sinusoidal hyperstable population of Figures 1B and 2B. The effects of the cyclical birth sequence are apparent in the waves that move through the population's age structure.

Actual and Hyperstable NRRs for the United States, 1960-89

The ability of NRRs produced by the fixed f, proportional ϕ, and constant A approaches to reproduce observed NRRs can be examined using the experience of the United States over the 1960-89 interval. The actual NRR values for 1960-89 were obtained [5] from Heuser (1976) and U.S. National Center for Health Statistics (1994). The births observed in the U.S. between 1920 and 1989

were taken from U.S. National Center for Health Statistics (1994, Table 1-1), and rescaled to show one birth in 1973. Using a base net maternity pattern based on the U.S. 1970, NRRs were calculated under our 3 hyperstable assumptions. Figure 4 shows that the 3 hyperstable NRRs provide fairly good estimates of observed behavior, even though U.S. net maternity did not follow any of those patterns. While the robustness of hyperstable estimation has yet to be examined in depth, this example suggests that hyperstable estimates may be applicable to many observed populations.

AN APPLICATION TO MODELING TRAJECTORIES TO STATIONARITY

Recently there has been considerable interest in the ultimate size of the world's population, and how it would vary under different scenarios of fertility decline (e.g. Bos et al, 1992). Trajectories to stationarity under alternative assumptions about future declines in births can be calculated using the double exponential model of equations (29)-(32). In equation (30), the birth trajectory depends on 2 parameters. Parameter b represents the rate of decline in the number of births, while parameter c is related to the initial NRR. Under the constant A assumption, equations (10) and (30) yield

$$c = b \ln R_0(0) / (e^{bA} - 1) \qquad (44)$$

The ultimate number of births can then be written

$$[1/(e^{bA} - 1)]$$

$$g(4) = R_0(0) \qquad (45)$$

Not surprisingly, the size of the ultimate birth cohort is smaller when the initial NRR is lower, the rate of decline in births is greater, and the length of a generation is longer. However, the double exponential hyperstable model states the relationship precisely, and makes it easier to compare the NRRs and the birth trajectories and ultimate levels that result from alternative assumptions.

Figure 5 shows the birth trajectories to stationarity for two populations with an initial NRR of 2.0 and A=27, but with different rates of decline in the number of births. For trajectory $g_1(t)$, $b_1 = .050$, and for trajectory $g_2(t)$, $b_2 = .025$. There is a very large difference in ultimate cohort size, with $g_1(4) = 1.27$ and $g_2(4) = 2.05$. While NRR(4) = 1 in both cases, net maternity drops much faster under the g_1 trajectory. Differences in NRRs based on the fixed f, proportional ϕ, and constant A assumptions are quite small.

SUMMARY AND CONCLUSION

Stable population models, based on fertility and mortality rates that do not change over time, are insufficiently flexible and realistic to capture the dynamics of populations with changing vital rates. Hyperstable models allow the rates to change, and relate a specified birth trajectory to a consistent set of net maternity schedules. Their potential usefulness in

demographic modeling and estimation appears to be substantial, as they replace the static equilibrium of classical stability with the dynamic equilibrium of hyperstability.

FOOTNOTES

1. The focus here differs from the stochastic ergodicity of age distributions in populations with Markovian vital rates, a subject investigated by Cohen (1976, 1977).

2. In most cases, there appear to be an infinite number of solutions for $\phi(x,t)$, and it is usually not difficult to find one that is demographically realistic. However, when g(t) is non-monotonic, the variance of $\phi(x,t)$ decreases when t approaches A years after an extremum. Exactly A years after an extremum of g(t), all net maternity is concentrated at age A.

3. The mean and variance of net maternity in the United States 1959-61, as given in Keyfitz and Flieger (1968, p153), were used as the parameters of a Wicksell (gamma) function in order to produce the initial net maternity function (cf. Keyfitz, 1977, p156-57).

4. Total Fertility Rates from Heuser (1976, Table 1) were changed into NRRs by (i) multiplying by 100/205 to produce Gross Reproduction Rates (GRRs) and (ii) incorporating mortality by multiplying those GRRs by the probability of surviving to age 30. Survival probabilities were taken from period life tables appearing in Schoen (1987).

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